Worcester County Mathematics League

Varsity Meet 4 - February 27, 2019

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 4 - February 27, 2019
Answer Key

Round 1 - Elementary Number Theory

1. 28
2. $12363_{(7)}$
3. 56

Round 2 - Algebra I

1. 53
2. $x = 6, 11$
3. $-1, 12$

Round 3 - Geometry

1. 142° or 142 degrees
2. 400π
3. $12 + 6\sqrt{3}$

Round 4 - Logs, Exponents, and Radicals

1. $n\sqrt[4]{2m^3n}$
2. $b = 3$
3. 352

Round 5 - Trigonometry

1. $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
2. $\tan(2) < \tan(3) < \tan(4) < \tan(1)$
3. $\dfrac{20 - 2\sqrt{5}}{45}$

Team Round

1. 6
2. $x = -18$ and $y = 12$
3. $\dfrac{\sqrt{11}}{2}$
4. $x = \dfrac{5 - 2a}{a - 1} = \dfrac{2a - 5}{1 - a}$ (no logs in answer)
5. $\dfrac{13}{12} = 1\dfrac{1}{12} = 1.08\overline{3}$
6. 64
7. 72
8. $\dfrac{11\pi}{14}$
9. $\dfrac{4}{3} = 1\dfrac{1}{3} = 1.\overline{3}$

Worcester County Mathematics League
Varsity Meet 4 - February 27, 2019
Round 1 - Elementary Number Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If $\operatorname{lcm}(a, b)$ represents the least common multiple of a and b, and $\operatorname{gcf}(a, b)$ represents the greatest common factor of a and b, what is the value of

$$\operatorname{lcm}(\operatorname{gcf}(\operatorname{gcf}(24, 16), \operatorname{lcm}(4, 5)), 14)$$

2. The following numbers are written in base seven. Evaluate and express your answer in base seven.

$$4251_{(7)} + 6463_{(7)} - 1351_{(7)}$$

3. Find the last two digits of 2^{508}.

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________$_{(7)}$

(3 pts) 3. ____________________

QSC, Bancroft, Worcester Academy

Worcester County Mathematics League
Varsity Meet 4 - February 27, 2019
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. The sum of 3 consecutive odd integers is 108 more than the least integer. What is the mean of the three integers?

2. Solve for x: $$\frac{x-3}{x-1} - \frac{x-2}{x+4} = \frac{1}{5}$$

3. Find all real solutions to the equation
$$(x-4)(x-5)(x-6)(x-7) = 1680.$$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. $x =$ ____________________

(3 pts) 3. ____________________

Bromfield, Algonquin, QSC

Worcester County Mathematics League

Varsity Meet 4 - February 27, 2019

Round 3 - Geometry

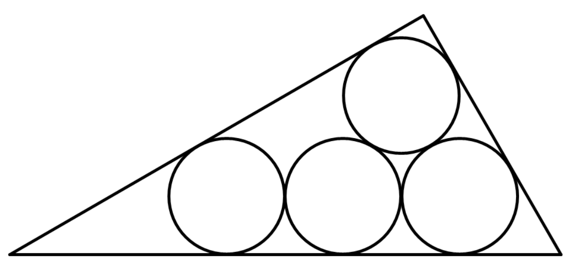
All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. The interior angles of an 8-sided convex polygon have measurements in degrees that are consecutive even integers. What is the measurement of the largest interior angle?

2. Two concentric circles have radii x and y with $x < y$. A line tangent to the inner circle at B intersects the outer circle at A and C. If $AC = 40$, find the area of the region between the two circles.

3. In the diagram below, four circles each of radius 1 are tangent to one another and to the sides of the triangle as shown. Find the perimeter of the triangle.



ANSWERS

(1 pt) 1. ______________________________ degrees

(2 pts) 2. ______________________________

(3 pts) 3. ______________________________

Northbridge, Tantasqua, AMSA

Worcester County Mathematics League

Varsity Meet 4 - February 27, 2019

Round 4 - Logs, Exponents, and Radicals

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Simplify.

$$\frac{\sqrt[4]{32m^7n^9}}{2mn}$$

2. Determine the value of b if

$$\log_b 135 = 2 + \log_b 3 + \log_b 5.$$

3. If the integers a, b, and c are all powers of 2 and

$$a^3 + b^4 = c^5,$$

what is the least possible value of $a + b + c$?

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. $b =$ ____________________

(3 pts) 3. ____________________

Bartlett, Auburn, Tahanto

Worcester County Mathematics League

Varsity Meet 4 - February 27, 2019

Round 5 - Trigonometry

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. For $0 \leq \theta < 2\pi$, find all solutions to
$$4\sin^2\theta + 5 = 8.$$

2. Order the following from least to greatest:
$$\tan(1), \tan(2), \tan(3), \tan(4)$$

3. Evaluate and express as a single fraction in simplest form.
$$\sin\left(2\arccos\left(\frac{2}{3}\right) - \arctan\left(-2\right)\right)$$

ANSWERS

(1 pt) 1. $\theta =$ ____________________

(2 pts) 2. ________ < ________ < ________ < ________

(3 pts) 3. ____________________

Shrewsbury, QSC, QSC

Worcester County Mathematics League
Varsity Meet 4 - February 27, 2019
Team Round

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Write $2019_{(10)}$ in the form $ABCDEF_{(4)}$ and find the product of B and D.

2. Solve for x and y:

$$\begin{cases} \frac{2}{3}x + \frac{5}{2}y = 18 \\ \frac{1}{2}x - \frac{1}{4}y = -12 \end{cases}$$

3. Find the area of a triangle whose sides are $\sqrt{3}$, 2, and $\sqrt{5}$.

4. If $5^x = 40$ and $\log 50 = a$, find x in terms of a. [There should be no logarithmic expressions in your answer.]

5. An acute angle θ has $\tan\theta = .4166\overline{6}$. Find $\sec\theta$.

6. If each letter in the subtraction problem below stands for one unique digit, find the value of B^C (B raised to the power of C):

$$\begin{array}{r} \text{A B C D C} \\ \text{- B E A A C} \\ \hline \text{B A D A D} \end{array}$$

7. How many four digit numbers have no repeat digits, do not contain zero, and have a sum of digits equal to 27?

8. Determine the value of $\arccos\left(\sin\left(\frac{9\pi}{7}\right)\right)$.

9. Consider a regular hexagon with an inscribed circle and a circumscribed circle. Find the ratio between the area of the larger circle and the area of the smaller circle.

Worcester County Mathematics League
Varsity Meet 4 - February 27, 2019
Team Round Answer Sheet

ANSWERS

1. ____________________

2. $x =$ __________ $y =$ __________

3. ____________________

4. $x =$ ____________________

5. ____________________

6. ____________________

7. ____________________

8. ____________________

9. ____________________

St. Peter-Marian, Algonquin, Algonquin, St. John's, Worcester Academy, Burncoat, QSC, QSC, QSC

Worcester County Mathematics League
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1. 53
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1. $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
2. $\tan(2) < \tan(3) < \tan(4) < \tan(1)$
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Team Round

1. 6
2. $x = -18$ and $y = 12$
3. $\frac{\sqrt{11}}{2}$
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6. 64
7. 72
8. $\frac{11\pi}{14}$
9. $\frac{4}{3} = 1\frac{1}{3} = 1.\overline{3}$

Round 1 - Elementary Number Theory

1. If $\text{lcm}(a, b)$ represents the least common multiple of a and b, and $\text{gcf}(a, b)$ represents the greatest common factor of a and b, what is the value of $\text{lcm}(\text{gcf}(\text{gcf}(24, 16), \text{lcm}(4, 5)), 14)$.

Solution: Work from the inside out.

$$\text{lcm}(\text{gcf}(\text{gcf}(24, 16), \text{lcm}(4, 5)), 14)$$
$$\text{lcm}(\text{gcf}(8, 20), 14)$$
$$\text{lcm}(4, 14)$$
$$\boxed{28}$$

2. The following numbers are written in base seven. Evaluate and express your answer in base seven.

$$4251_{(7)} + 6463_{(7)} - 1351_{(7)}$$

Solution: We start by adding the first two numbers, $4251_{(7)} + 6463_{(7)}$. In each column, we sum the digits and carry the one every seven that we count. This means we have some carry-over from three of the four columns.

$$\begin{array}{rr} & 4\,2\,5\,1_{(7)} \\ + & 6\,4\,6\,3_{(7)} \\ \hline & 1\,4\,0\,4\,4_{(7)} \end{array}$$

Then we need to subtract the third number.

$$\begin{array}{rr} & 1\,4\,0\,4\,4_{(7)} \\ + & 1\,3\,5\,1_{(7)} \\ \hline & ?\,?\,?\,?\,?_{(7)} \end{array}$$

In this situation, we need to borrow a couple of times, and even borrow two columns away in order to subtract in the second column.

$$\begin{array}{rccccc} & 1 & \cancel{4}3 & \cancel{0}\cancel{7}6 & \cancel{4}11 & 4_{(7)} \\ + & & 1 & 3 & 5 & 1_{(7)} \\ \hline & 1 & 2 & 3 & 6 & 3_{(7)} \end{array}$$

The answer is $\boxed{12363_{(7)}}$.

3. Find the last two digits of 2^{508}.

Solution: Looking for a pattern, we find one (although it takes a short while). Realizing that from one value to the next of 2^n, the last two digits are only dependent upon the previous value's last two digits. In the table below we find that the last two digits of 2^n for $2 \leq n \leq 21$ are equivalent to the last two digits of 2^n for $22 \leq n \leq 41$.

n	2^n	n	2^n
1	2		
2	4	22	4194304
3	8	23	8388608
4	16	24	16777216
5	32	25	33554432
6	64	26	67108864
7	128	27	134217728
8	256	28	268435456
9	512	29	536870912
10	1024	30	1073741824
11	2048	31	2147483648
12	4096	32	4294967296
13	8192	33	8589934592
14	16384	34	17179869184
15	32768	35	34359738368
16	65536	36	68719476736
17	131072	37	137438953472
18	262144	38	274877906944
19	524288	39	549755813888
20	1048576	40	1099511627776
21	2097152	41	2199023255552

Thinking forward, the last two digits of 2^{21}, 2^{41}, ..., 2^{481}, and 2^{501} will be the same. That means the last two digits of 2^{508} will be equal to the last two digits of 2^{28} - $\boxed{56}$.

Round 2 - Algebra I

1. The sum of 3 consecutive odd integers is 108 more than the least integer. What is the mean of the three integers?

Solution: Let the three odd integers be represented by n, $n+2$, and $n+4$. If their sum is 108 more than the least integer, we know that

$$n + (n+2) + (n+4) = 108 + n$$

and are looking to find the average of the three integers which will be

$$\frac{n + (n+2) + (n+4)}{3} = \frac{3n+6}{3} = n+2,$$

otherwise known as the second of the three odd integers. Solving the first equation for n:

$$3n + 6 = 108 - n$$

$$2n = 102$$

$$n = 51$$

The mean of the integers, or the second of the odd integers, is $\boxed{53}$.

2. Solve for x: $\quad \dfrac{x-3}{x-1} - \dfrac{x-2}{x+4} = \dfrac{1}{5}$

Solution: First we note that $x \neq 1$ or -4. Then, multiplying through the equation by the common denominator, $5(x-1)(x+4)$, we can start to solve.

$$\begin{aligned}
\frac{x-3}{x-1} - \frac{x-2}{x+4} &= \frac{1}{5} \\
5(x+4)(x-3) - 5(x-2)(x-1) &= (x-1)(x+4) \\
5(x^2 + x - 12) - 5(x^2 - 3x + 2) &= x^2 + 3x - 4 \\
5x^2 + 5x - 60 - 5x^2 + 15x - 10 &= x^2 + 3x - 4 \\
0 &= x^2 - 17x + 66 \\
0 &= (x-6)(x-11)
\end{aligned}$$

By the Zero Product Property, our solutions are $\boxed{\text{6 and 11}}$. Neither is extraneous.

3. Find all real solutions to the equation

$$(x-4)(x-5)(x-6)(x-7) = 1680.$$

Solution: The question can be interpreted as "Find four numbers that are all one apart whose product is 1680." Factoring 1680, we find that $1680 = (168)(10) = (8)(21)(10) = (8)(7)(3)(5)(2) = (8)(7)(6)(5)$. Comparing this to the four factors, we find $x = 12$ to be one of our solutions. We also know that $1680 = (8)(7)(6)(5) = (-5)(-6)(-7)(-8)$. Comparing this to the four factors, we find $x = -1$ to be another solution.

But what about other solutions? Any number $x > 12$ will generate a product too large, as will any number $x < -1$, and numbers on the interval $(-1, 12)$ will cause the product to be too small. Our two solutions: $\boxed{-1 \text{ and } 12}$.

x	x-4	x-5	x-6	x-7	Product
-1	-5	-6	-7	-8	**1680**
0	-4	-5	-6	-7	**840**
1	-3	-4	-5	-6	**360**
2	-2	-3	-4	-5	**120**
3	-1	-2	-3	-4	**24**
4	0	-1	-2	-3	**0**
5	1	0	-1	-2	**0**
6	2	1	0	-1	**0**
7	3	2	1	0	**0**
8	4	3	2	1	**24**
9	5	4	3	2	**120**
10	6	5	4	3	**360**
11	7	6	5	4	**840**
12	8	7	6	5	**1680**

How to prove there are only two solutions? Show that the expression $(x-4)(x-5)(x-6)(x-7) - 1680$ can be written as the product of two linear factors of $(x-12)$ and $(x+1)$ and one irreducible quadratic factor, leading to two complex roots. Or consider the end behavior of the function $f(x) = (x-4)(x-5)(x-6)(x-7)$ shifted down 1680 units. Or follow me down this rabbit hole.

Alternative Solution 1: Notice the symmetry of the factors about the value 5.5. Let $x = m + \frac{11}{2}$ and the equation $(x-4)(x-5)(x-6)(x-7) = 1680$ changes to

$$\left(m+\frac{3}{2}\right)\left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right) = 1680$$
$$\left(m^2-\frac{9}{4}\right)\left(m^2-\frac{1}{4}\right) = 1680$$

Notice the symmetry of the factors about the value $\frac{5}{4}$. Let $m^2 = p + \frac{5}{4}$ and the equation changes to

$$\left(m^2-\frac{9}{4}\right)\left(m^2-\frac{1}{4}\right) = 1680$$
$$(p-1)(p+1) = 1680$$

$$p^2 - 1 = 1680$$
$$p^2 = 1681$$
$$p = \pm 41$$
$$m^2 - \frac{5}{4} = \pm 41$$
$$m = \pm\sqrt{\pm 41 + \frac{5}{4}}$$
$$x - \frac{11}{2} = \pm\sqrt{\pm 41 + \frac{5}{4}}$$
$$x = \pm\sqrt{\pm 41 + \frac{5}{4}} + \frac{11}{2}$$

Since 41 must be positive, our values for x are

$$x = \pm\sqrt{41 + \frac{5}{4}} + \frac{11}{2}$$
$$x = \pm\sqrt{\frac{169}{4}} + \frac{11}{2}$$
$$x = \pm\frac{13}{2} + \frac{11}{2} = \boxed{-1 \text{ or } 12}$$

Alternative Solution 2: Rearrange and multiply:

$$(x-4)(x-5)(x-6)(x-7) = 1680$$
$$(x-4)(x-7)(x-5)(x-6) = 1680$$
$$(x^2 - 11x + 28)(x^2 - 11x + 30) = 1680$$

Let $m = x^2 - 11x + 28$:

$$(m)(m+2) = 1680$$
$$m^2 + 2m = 1680$$
$$m^2 + 2m + 1 = 1681$$
$$(m+1)^2 = 41^2$$
$$m + 1 = \pm 41$$

This leaves us with $m = 40$ or $m = -42$. For the first solution:

$$x^2 - 11x + 28 = 40$$
$$x^2 - 11x - 12 = 0$$
$$(x-12)(x+1) = 0$$

Two real solutions: $\boxed{-1 \text{ or } 12}$. For the next solution:

$$x^2 - 11x + 28 = -42$$
$$x^2 - 11x + 70 = 0$$

In solving this, $b^2 - 4ac = 121 - 280 = -159$, meaning the other two solutions are complex.

Round 3 - Geometry

1. The interior angles of an 8-sided convex polygon have measurements in degrees that are consecutive even integers. What is the measurement of the largest interior angle?

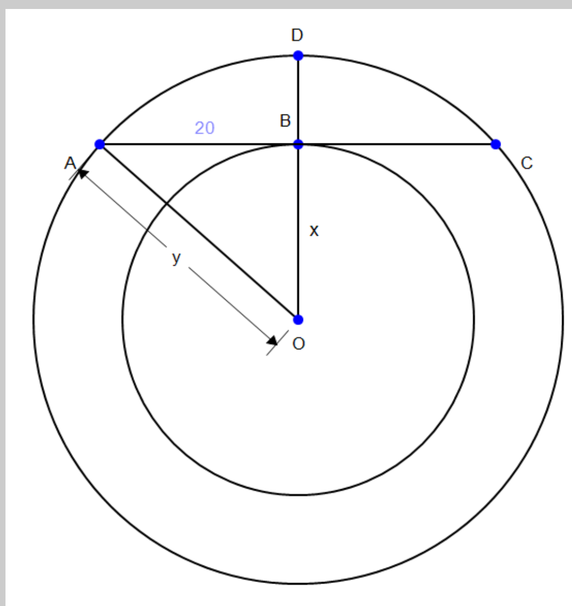
Solution: The sum of the angles in a convex octagon is $6 \cdot 180° = 1080°$. Knowing the average angle measurement is $1080° \div 8 = 135°$ we know the angle measures are symmetrical about 135°, namely: 128°, 130°, 132°, 134°, 136°, 138°, 140°, $\boxed{142°}$. We can also find this through algebra:

$$\begin{aligned}(x)° + (x+2)° + (x+4)° + (x+6)° + (x+8)° + (x+10)° + (x+12)° + (x+14)° &= 1080° \\ (8x+56)° &= 1080° \\ (8x)° &= 1024° \\ x° &= 128°\end{aligned}$$

The largest angle? $(x+14)° = 142°$

2. Two concentric circles have radii x and y with $x < y$. A line tangent to the inner circle at B intersects the outer circle at A and C. If $AC = 40$, find the area of the region between the two circles.

Solution: First, draw a diagram.



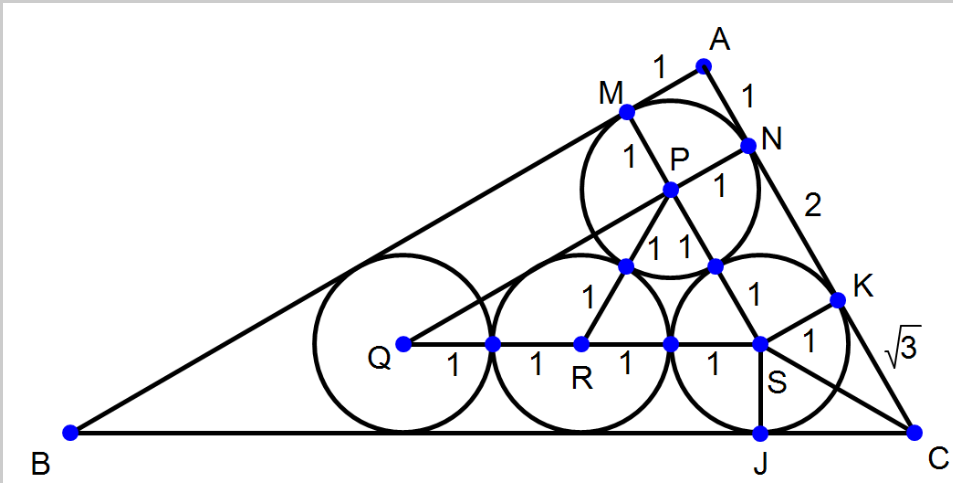
Since $AB = 20$, we can establish through the triangle shown that $x^2 + 20^2 = y^2$ or $y^2 - x^2 = 400$. Multiplying through by π leaves us with

$$\pi y^2 - \pi x^2 = \boxed{400\pi}$$

which is the difference between the area of the larger circle (πy^2) and the smaller circle (πx^2).

3. In the diagram below, four circles each of radius 1 are tangent to one another and to the sides of the triangle as shown. Find the perimeter of the triangle.

Solution: Consider the following diagram.



Note that Q, R, and S are collinear, and that the sides of ΔPQS are parallel to the sides of ΔABC, meaning that $\Delta PQS \sim \Delta ABC$ and their perimeters are proportional.

Since R is equidistant from Q, R, and S, and ΔPQS is inscribed in a semicircle with diameter $\overline{QS}$, then $\measuredangle QPS = 90°$. Additionally, since ΔPRS is equilateral, $\measuredangle PSQ = 60°$, $\measuredangle SQP = 30°$, and ΔPQS is a right triangle. The perimeter of ΔPQS is $4 + 2 + \sqrt{(4^2 - 2^2)} = 6 + \sqrt{12} = 6 + 2\sqrt{3}$.

To find the scale factor of the triangles, which are both $30° - 60° - 90°$ right triangles, we find the scale factor between PS and AC. We find $AN = 1$ (since $AMPN$ is a square with side length 1), $NK = 2$ (since $PNKS$ is a rectangle), and $KC = \sqrt{3}$ (since ΔKSC is a $30° - 60° - 90°$ right triangle).

Since $\frac{AC}{PS} = \frac{3+\sqrt{3}}{2}$ we multiply this scale factor by the perimeter of ΔPSQ to find the perimeter of the larger triangle.

$$\frac{3+\sqrt{3}}{2} \cdot (6 + 2\sqrt{3}) = (3 + \sqrt{3})^2 = \boxed{12 + 6\sqrt{3}}$$

Round 4 - Logs, Exponents, and Radicals

1. Simplify.

$$\frac{\sqrt[4]{32m^7n^9}}{2mn}$$

Solution:

$$\frac{\sqrt[4]{32m^7n^9}}{2mn} = \frac{\sqrt[4]{2^5m^7n^9}}{2mn} = \frac{2mn^2\sqrt[4]{2m^3n}}{2mn} = \boxed{n\sqrt[4]{2m^3n}}$$

2. Determine the value of b if

$$\log_b 135 = 2 + \log_b 3 + \log_b 5.$$

Solution: Combining the terms on the right side, we find

$$\log_b 135 = 2 + \log_b 15$$

and raising the base b to both sides, we find

$$b^{\log_b 135} = b^{2+\log_b 15}$$

$$135 = b^2 \cdot 15$$

$$9 = b^2$$

$$\pm 3 = b$$

Since the base must be positive, $\boxed{b = 3}$.

Alternatively:

$$\log_b 135 = 2 + \log_b 15$$

$$\log_b 135 = \log_b b^2 + \log_b 15$$

$$\log_b 135 = \log_b 15b^2$$

$$135 = 15b^2$$

$$\pm 3 = b$$

Or:

$$\log_b 135 = 2 + \log_b 15$$

$$\log_b 135 - \log_b 15 = 2$$

$$\log_b 9 = 2$$

$$9 = b^2$$

$$\pm 3 = b$$

3. If the integers a, b, and c are all powers of 2 and

$$a^3 + b^4 = c^5,$$

what is the least possible value of $a + b + c$?

Solution:

Let $a = 2^m$, $b = 2^n$, and $c = 2^p$. Remember that these are all non-negative integers so m, n, and p must be positive integer values. Then

$$2^{3m} + 2^{4n} = 2^{5p}$$

and the first two terms must be equal; if they were not equal (say, 128 and 64), then their sum would not also be a power of two. We find that $3m = 4n \Rightarrow \frac{m}{n} = \frac{4}{3}$. We also find that

$$2^{3m} + 2^{4n} = 2^{5p}$$

$$2^{3m} + 2^{3m} = 2^{5p}$$

$$2 \cdot 2^{3m} = 2^{5p}$$

$$\cdot 2^{3m+1} = 2^{5p}$$

$$3m + 1 = 5p$$

$$p = \frac{3m+1}{5}$$

Recall our goal is the find the lowest value of $a + b + c$, which means we trying to find the lowest value of 2^m, 2^n, and 2^p, which means we are trying to find the lowest values of m, n, and p. Since m, n, and p are all positive integer values, let's try $m = 4$, $n = 3$. We find that p would need to be $p = \frac{3 \cdot 4+1}{5} = \frac{13}{5}$ so this doesn't work (p must be an integer).

Trying the next lowest possilbe integer values for m, n, and p, we try $m = 8$, $n = 6$ and $p = \frac{3 \cdot 8+1}{5} = 5$. They are all integers, and any other trio of integers would cause $a+b+c$ to be larger. Therefore, the least possible value of $a + b + c$ is

$$a + b + c = 2^m + 2^n + 2^p = 2^8 + 2^6 + 2^5 = 256 + 64 + 32 = \boxed{352}$$

Round 5 - Trigonometry

1. For $0 \le \theta < 2\pi$, find all solutions to

$$4\sin^2\theta + 5 = 8.$$

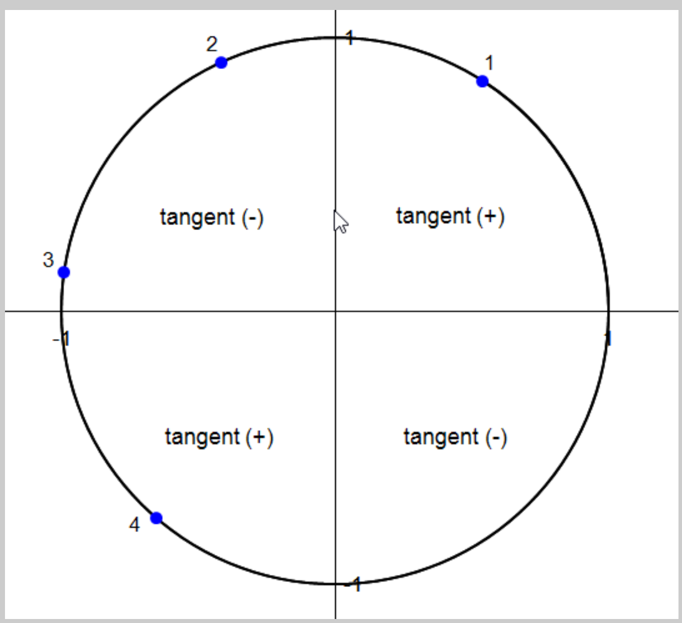
Solution: Solving first for $\sin\theta$:

$$\begin{aligned} 4\sin^2\theta + 5 &= 8 \\ 4\sin^2\theta &= 3 \\ \sin^2\theta &= \frac{3}{4} \\ \sin\theta &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

The four solutions are $\boxed{\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$.

2. Order the following from least to greatest: $\tan(1), \tan(2), \tan(3), \tan(4)$.

Solution: The diagram below shows the approximate location of 1, 2, 3, and 4 radians as well as the signage of the tangent function for each quadrant of the unit circle.



From this, we know that $\tan(2)$ and $\tan(3)$ are less than $\tan(1)$ and $\tan(4)$. Also, since the tangent function is always increasing as $\theta \to \infty$ (aside from breaks at odd integer multiples of $\frac{\pi}{2}$), we know that $\tan(2)$ will be less than $\tan(3)$ which will be less than $\tan(4)$. Since 3 radians is not quite a half-revolution of the unit circle, 1 radian is further along the unit circle in a positive quadrant than is 4 radians, meaning that $\tan(1)$ is greater than $\tan(4)$. This leaves us with the following inequality: $\boxed{\tan(2) < \tan(3) < \tan(4) < \tan(1)}$.

3. Evaluate and express as a single fraction in simplest form.

$$\sin\left(2\arccos\left(\frac{2}{3}\right) - \arctan\left(-2\right)\right)$$

Solution: Recognize that this is essentially the difference formula for sine. Let

$$\arccos\left(\frac{2}{3}\right) = \alpha \quad \text{and} \quad \arctan\left(-2\right) = \beta$$

which means

$$\cos\alpha = \frac{2}{3} \quad \text{and} \quad \tan\beta = -2$$

. We can now rewrite our original expression as

$$\sin(2\alpha - \beta) = \sin 2\alpha \cos\beta - \cos 2\alpha \sin\beta = 2\sin\alpha\cos\alpha\cos\beta - [2\cos^2\alpha - 1]\sin\beta$$

Filling in for what we know so far:

$$\sin(2\alpha - \beta) = 2\sin\alpha\left(\frac{2}{3}\right)\cos\beta - [2\left(\frac{2}{3}\right)^2 - 1]\sin\beta = \frac{4}{3}\sin\alpha\cos\beta + \frac{1}{9}\sin\beta$$

Since $\cos\alpha = \frac{2}{3} = \frac{x}{r}$ and α is in the first quadrant, then $\sin\alpha = \frac{y}{r} = \frac{\sqrt{5}}{3}$. Additionally, since $\tan\beta = \frac{y}{x} = \frac{-2}{1}$ and β is in the fourth quadrant, then $\sin\beta = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$ and $\cos\beta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$. We now substitute and simplify:

$$\sin(2\alpha - \beta) = \frac{4}{3}\cdot\frac{\sqrt{5}}{3}\cdot\frac{\sqrt{5}}{5} + \frac{1}{9}\cdot\frac{-2\sqrt{5}}{5} = \boxed{\frac{20 - 2\sqrt{5}}{45}}$$

Team Round

1. Write $2019_{(10)}$ in the form $ABCDEF_{(4)}$ and find the product of B and D.

Solution: The first digit, A, represents how many 4^5s, or 1024s, are in 2019 [1]. $2019 - 1024 = 995$, so we look at the next digit.

The second digit, B, represents how many 4^4s, or 256s, are in 2019 after subtracting the number of 1024s from 2019 [3]. $995 - 768 = 227$, so we look at the next digit.

The third digit, C, represents how many 4^3s, or 64s, are in 2019 after subtracting the number of 1024s and 256s from 2019 [3]. $227 - 192 = 35$, so we look at the next digit.

The fourth digit, D, represents how many 4^2s, or 16s, are in 2019 after subtracting the number of 1024s and 256s and 64s from 2019 [2]. $35 - 32 = 3$, so we look at the next digit.

The fifth digit, E, represents how many 4^1s, or 4s, are in 2019 after subtracting the number of 1024s and 256s and 64s and 16s from 2019 [0]. $3 - 0 = 3$, so look at the final digit.

The final digit, F, represents how many 4^0s, or 1s, are in 2019 after subtracting the number of 1024s and 256s and 64s and 16s and 4s from 2019 [3]. $3 - 3 = 0$, so we're done.

$2019_{(10)}$ is equivalent to $133203_{(4)}$. Multiplying the second digit by the fourth digit is 6.

Alternative Solution: Utilize the division process. Begin with dividing 2019 by 4 to get 504 R [3]. Then divide 504 by 4 to get 126 R [0]. Then divide 126 by 4 to get 31 R [2]. Then divide 31 by 4 to get 7 R [3]. Then divide 7 by 4 to get 1 R [3]. Then divide 1 by 4 to get 0 R [1]. Place these remainders in reverse order to get $133203_{(4)}$.

2. Solve for x and y:

$$\begin{cases} \frac{2}{3}x + \frac{5}{2}y = 18 \\ \frac{1}{2}x - \frac{1}{4}y = -12 \end{cases}$$

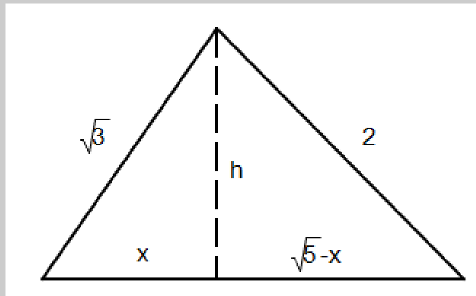
Solution: Clear the denominators by multiplying through by 6 and by 4:

$$\begin{cases} 4x + 15y = 108 \\ 2x - y = -48 \end{cases}$$

Multiply the bottom equation by -2 and add the equations together to get $17y = 204$. From this we find that $\boxed{y = 12}$. Plugging back in to the second augmented equation gives us $2x - 12 = -48 \Rightarrow 2x = -36$ and $\boxed{x = -18}$.

3. Find the area of a triangle whose sides are $\sqrt{3}$, 2, and $\sqrt{5}$.

Solution: Draw a diagram with $\sqrt{5}$ as the base of the triangle since it is the longest side of the three. Drop an altitude and label each segment.



Since we have two right triangles, we establish

$$x^2 + h^2 = 3 \qquad\qquad (\sqrt{5} - x)^2 + h^2 = 4$$
$$h^2 = 3 - x^2 \qquad\qquad h^2 = 4 - (\sqrt{5} - x)^2$$

Solving for x by the transitive property:

$$3 - x^2 = 4 - 5 + 2\sqrt{5}x - x^2$$
$$4 = 2\sqrt{5}x$$
$$\frac{2}{\sqrt{5}} = x$$

Solving for h:

$$\left(\frac{2}{\sqrt{5}}\right)^2 + h^2 = 3$$
$$\frac{4}{5} + h^2 = 3$$
$$h^2 = \frac{11}{5}$$
$$h = \sqrt{\frac{11}{5}}$$

With a base of $\sqrt{5}$ and a height of $\sqrt{\frac{11}{5}}$, the area of the triangle is $\frac{1}{2} \cdot \sqrt{5} \cdot \sqrt{\frac{11}{5}} = \boxed{\frac{\sqrt{11}}{2}}$.

4. If $5^x = 40$ and $\log 50 = a$, find x in terms of a. [There should be no logarithmic expressions in your answer.]

Solution: Isolating x to start, we know that $\log_5 40 = x$ and that

$$x = \frac{\log 40}{\log 5}.$$

Since $\log 50 = \log 5 + \log 10 = \log 5 + 1 = a$ we can rewrite x as

$$x = \frac{\log 40}{a-1}.$$

Since $\log 40 = \log 4 + \log 10 = 2\log 2 + 1 = 2(\log 10 - \log 5) + 1 = 2(1-(a-1)) + 1$, we can rewrite x as

$$x = \frac{2(1-(a-1))+1}{a-1} = \frac{2(2-a)+1}{a-1} = \boxed{\frac{5-2a}{a-1}}.$$

NB: This can also be expressed as $\dfrac{2a-5}{1-a}$.

5. An acute angle θ has $\tan\theta = .4166\overline{6}$. Find $\sec\theta$.

Solution: First, convert $0.4166\overline{6}$ into a proper fraction:

$$\begin{aligned} N &= .41\overline{6} \\ 1000N &= 416.\overline{6} \\ 100N &= 41.\overline{6} \end{aligned}$$

Subtracting the third equation from the second gives us $900N = 375 \Rightarrow N = \frac{375}{900} = \frac{5}{12}$. Since $\tan\theta = \frac{5}{12}$ and $tan^2\theta + 1 = \sec^2\theta$,

$$\sec\theta = \sqrt{\tan^2\theta + 1} = \sqrt{\frac{25}{144} + 1} = \sqrt{\frac{169}{144}} = \boxed{\frac{13}{12}}.$$

Alternative Solution: Using x, y, and r and $x^2 + y^2 = r^2$, since $\tan\theta = \frac{y}{x} = \frac{5}{12}$, $r = 13$ and $\sec\theta = \frac{r}{x} = \frac{13}{12}$.

6. If each letter in the subtraction problem below stands for one unique digit, find the value of B^C (B raised to the power of C):

$$\begin{array}{r} \text{A B C D C} \\ \text{- B E A A C} \\ \hline \text{B A D A D} \end{array}$$

Solution: Looking at the ones column, $C - C = D$, which means that $D = 0$. Looking at the tens column, we now see that $0 - A = A$, and since $A \neq 0$, we carry the 1 from the hundreds column to find that $10 - A = A$ and therefore $A = 5$. Our problem can now be written as

$$\begin{array}{r} \text{5 B C 0 C} \\ \text{- B E 5 5 C} \\ \hline \text{B 5 0 5 0} \end{array}$$

In the hundreds column, we see that $C - 5 = 0$, but $C \neq 5$ since all the digits are unique, but recall that we borrowed a 1 from C to subtract in the tens column, so $C = 6$ and we have

$$\begin{array}{r} 52606 \\ -27556 \\ \hline 25050 \end{array}$$

In the ten-thousands column, we see that $5 - B = B$ which isn't possible unless a 1 needs to be borrowed from that column for the subtraction in the thousands column, meaning $B = 2$ and $E = 7$.

$$\begin{array}{r} 52606 \\ -27556 \\ \hline 25050 \end{array}$$

Now we can find that $B^C = 2^6 = \boxed{64}$.

7. How many four digit numbers have no repeat digits, do not contain zero, and have a sum of digits equal to 27?

Solution: For four non-zero digits to sum to 27, there are only three possibilities:

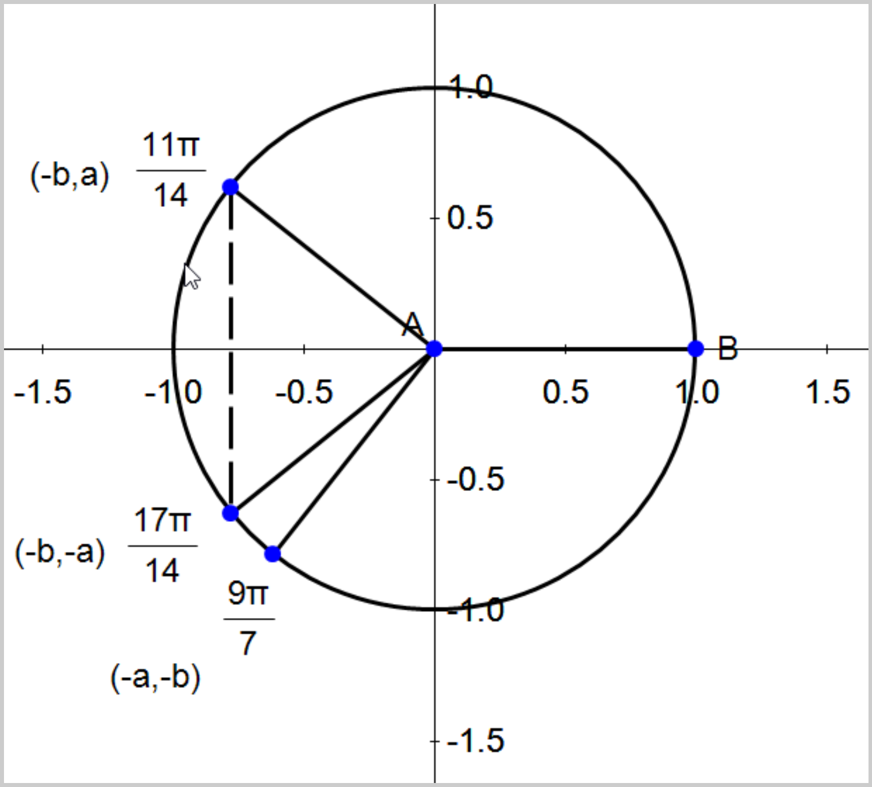
- $9 + 8 + 7 + 3$
- $9 + 8 + 6 + 5$
- $9 + 7 + 6 + 5$

Each of these groups of four digits can be arranged in $4! = 24$ ways, and with three groups, there are $24 * 3 = \boxed{72}$ four-digit numbers that fit the criteria.

8. Determine the value of $\arccos\left(\sin\left(\frac{9\pi}{7}\right)\right)$.

Solution: There's two approaches here.

Long Solution: The question being asked by the expression is "What is the angle in the second quadrant where cosine has the same value as sine of $\frac{9\pi}{7}$?" Consider the unit circle diagram below.



Let the point on the unit circle at $\frac{9\pi}{7}$ have coordinates $(-a,-b)$ where a and b are positive numbers between 0 and 1 and $a^2+b^2=1$. At this point, $-b$ is the value of $\sin(\frac{9\pi}{7})$, but we want it to be the value of cosine, so if we find the 3rd quadrant complement (which is something I just made up) of $\frac{9\pi}{7}$, we find it by determining how far away $\frac{9\pi}{7}$ is from $\frac{3\pi}{2}$ and then adding that value to π:

$$\frac{3\pi}{2}-\frac{9\pi}{7}=\frac{21\pi-18\pi}{14}=\frac{3\pi}{14}$$

$$\frac{3\pi}{14}+\pi=\frac{17\pi}{14}$$

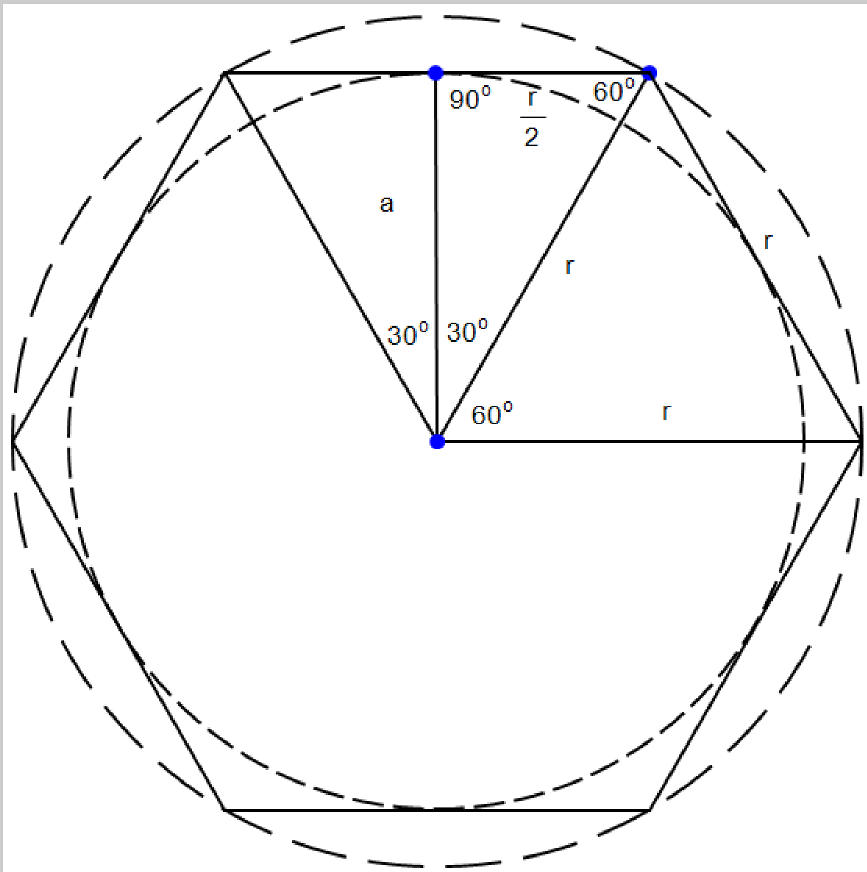
The coordinates on the unit circle at $\frac{17\pi}{14}$ are now $(-b,-a)$. Since the inverse cosine function only returns angles in the first or second quadrants, we need to identify the angle in the 2nd quadrant which has $-b$ as its x-coordinate, so we need to go back and **subtract** $\frac{3\pi}{14}$ from π to arrive at $\boxed{\frac{11\pi}{14}}$.

Short Solution: Since $\sin(\theta)=\cos\left(\frac{\pi}{2}-\theta\right)$, $\cos(-\theta)=\cos(\theta)$, and $\arccos(\cos\theta)=\theta$ for $\theta\in[0,\pi]$ we can rewrite the original expression and simplify

$$\arccos\left(\sin\left(\frac{9\pi}{7}\right)\right)=\arccos\left(\cos\left(\frac{\pi}{2}-\frac{9\pi}{7}\right)\right)=\arccos\left(\cos\left(\frac{-11\pi}{14}\right)\right)=\arccos\left(\cos\left(\frac{11\pi}{14}\right)\right)=\boxed{\frac{11\pi}{14}}.$$

9. Consider a regular hexagon with an inscribed circle and a circumscribed circle. Find the ratio between the area of the larger circle and the area of the smaller circle.

Solution: Let r be the radius of the hexagon, which is also the radius of the circumscribed circle (external) and let a be the apothem of the hexagon, which is also the radius of the inscribed circle (internal). We start by finding a in terms of r. Consider the diagram below.



Since the rix radii of a hexagon split the hexagon into equilateral triangles, we can construct a right triangle with legs of a, $\frac{r}{2}$. amd r. We know that since $a^2 + (\frac{r}{2})^2 = r^2$, then $a^2 = r^2 - (\frac{r}{2})^2$.

We are looking for the ratio of the larger area to the smaller area:

$$\frac{\text{larger area}}{\text{smaller area}} = \frac{\pi r^2}{\pi a^2} = \frac{r^2}{r^2 - \frac{r^2}{4}} = \frac{1}{1 - \frac{1}{4}} = \boxed{\frac{4}{3} = 1\frac{1}{3} = 1.\overline{3}}$$